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REPRESENTATIONS OF DIRICHLET AVERAGES OF GENERALIZED MITTAG-LEFFLER FUNCTION VIA FRACTIONAL INTEGRALS AND SPECIAL FUNCTIONS

Anatoly A. Kilbas [1], Anitha Kattuveettil [2]

Abstract

The paper is devoted to the investigation of Dirichlet averages of the generalized Mittag-Leffler function. Representation for such constructions in terms of the Riemann-Liouville fractional integrals and of the hypergeometric functions of many variables are established in two- and multi-dimensional cases. Special cases when the above Dirichlet averages coincide with the Mittag-Leffler function and hypergeometric functions of one and many variables are considered.

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1. Introduction

This paper is devoted to investigation of the generalized Mittag-Leffler function

$$E^{\gamma}_{\alpha,\delta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k \; z^k}{\Gamma(\alpha k + \delta) k!}, \tag{1}$$

with complex $\alpha,\ \delta,\ \gamma \in \mathbb{C},\ \ \Re(\alpha) > 0$. Here $(\gamma)_k$ is the Pochhammer symbol defined for $\gamma \in \mathbb{C}$ and integer non-negative $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \mathbb{N} = \{1, 2, \cdots\}$, by

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1) \quad (k \in \mathbb{N}). \tag{2}$$

Such a function, introduced by Prabhakar [15] is an entire function of z of order $[\Re(\alpha)]^{-1}$. Properties of this function were investigated by Prabhaker [15] and Kilbas et al [8], [9]. In particular, when $\gamma = 1$, (1) coincides with the classical Mittag-Leffler function $E_{\alpha,\beta}(z)$, [5]:

$$E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)k!} \quad (\alpha, \beta \in \mathbb{C},\ \Re(\alpha) > 0). \tag{3}$$

This function plays an important role in the theory of fractional differential equations and its applications to diffusion problems arising in physics, mechanics and other applied problems, for example, see [6], [7], [11], [13], [14]. We also note that various generalizations and modifications of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ are arisen in fractional calculus and the theory of integral transforms and special functions, see for example [12], [16].

When $\alpha = 1$, (1) coincides with the Kummer hypergeometric function ${}_1F_1$ (see in [4]) apart from a constant factor $[\Gamma(\delta)]^{-1}$:

$$E^{\gamma}_{1,\delta}(z) = \frac{1}{\Gamma(\delta)}\ {}_1F_1(\gamma; \delta; z) = \frac{1}{\Gamma(\delta)} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{(\delta)_k} \frac{z^k}{k!} \quad (\gamma, \delta \in \mathbb{C},\ \Re(\delta) > 0). \tag{4}$$

Our paper is devoted to the study of the Dirichlet averages of the generalized Mittag-Leffler function (1) in the forms

$$M^{\gamma}_{\alpha,\delta}(\beta, \beta'; x, y) = \int_{E_1} E^{\gamma}_{\alpha,\delta}(u \circ z) d\mu_{\beta,\beta'}(u), \tag{5}$$

$$(\alpha, \delta, \gamma, \beta, \beta' \in \mathbb{C},\ \Re(\alpha) > 0;\ \Re(\beta) > 0,\ \Re(\beta') > 0;\ x, y \in \mathbb{R}),$$

and

$$M^{\gamma,\alpha}_{\beta,\delta}(b_1, \cdots, b_n; 1 - z_1, \cdots, 1 - z_n) = \int_{E_{n-1}} E^{\gamma}_{\beta,\delta}\left((1 - u \circ z)^{\alpha}\right) d\mu_b(u) \tag{6}$$

$$(\alpha,\ \beta,\ \delta,\ \gamma \in \mathbb{C},\ \Re(\beta) > 0;\ \ b_i, z_i \in \mathbb{C},\ \Re(b_i) > 0,\ i = 1, \cdots, n),$$

and two of their modifications. Here E_{n-1} is the standard simplex in $\mathbb{R}^{n-1}$, $n \geq 2$:

$$E_{n-1} = \{(u_1, u_2,\ \cdots, u_{n-1}):\ u_1 \geq 0, \cdots, u_{n-1} \geq 0,\ u_1 + \cdots + u_{n-1} \leq 1\}, \tag{7}$$

$$u \circ z = \sum_{i=1}^{n-1} u_i z_i + (1 - u_1 - \cdots - u_{n-1}) z_n, \tag{8}$$

and $d\mu_b$ is the so-called Dirichlet measure defined by

$$d\mu_b(u) = \frac{1}{B(b)}\, u_1^{b_1-1} \cdots u_{n-1}^{b_{n-1}-1}(1-u_1-\cdots-u_{n-1})^{b_n-1}du_1\cdots du_{n-1}, \quad (9)$$

where

$$B(b) = \frac{\Gamma(b_1)\cdots\Gamma(b_k)}{\Gamma(b_1+\cdots+b_k)} \quad (\Re(b_j)>0;\ j=1,\cdots,n), \quad (10)$$

$\Gamma(\cdot)$ being the gamma function. In particular, for $n=2$, $d\mu_{\beta,\beta'}(u)$ in (5) is given by

$$d\mu_{\beta,\beta'}(u) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}u^{\beta-1}(1-u)^{\beta'-1}. \quad (11)$$

The general Dirichlet average of a function $f(z)=f(z_1,\cdots,z_n)$ was defined by Carlson [2] in the form

$$F(b,z) = \int_{E_{n-1}} f(u\circ z)d\mu_b(u), \quad (12)$$

where $d\mu_b$ is defined by (9). For $n=1$, $F(b,z)=f(z)$. Carlson [2] investigated the average (12) for $f(z)=z^k$ with any real $k\in\mathbb{R}$ in the form

$$R_k(b,z) = \int_{E_{n-1}} (u\circ z)^k d\mu_b(u). \quad (13)$$

In particular if $n=2$, Carlson [2], [3] proved that

$$R_k(\beta,\beta';x,y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\int_0^1 [ux+(1-u)y]^k\ u^{\beta-1}(1-u)^{\beta'-1}du, \quad (14)$$

where $\beta,\ \beta'\in\mathbb{C}$ are complex numbers with positive real parts $\Re(\beta)>0$, $\Re(\beta')>0$, and $x,\ y\in\mathbb{R}$.

We prove the representation for (5) and (6) in terms of the Riemann-Liouville fractional integral of order $\alpha\in\mathbb{C}$, $\Re(\alpha)>0$, [16]:

$$[I^\alpha_{a+}f](x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt \quad (x>a,\ a\in\mathbb{R}), \quad (15)$$

and of the special hypergeometric function of many variables known as the Srivastava-Daoust function [18]:

$$F^{A:\ B';\cdots;B^{(n)}}_{C:\ D';\cdots;D^{(n)}}\left[\begin{matrix}[(a):\ \theta',\cdots,\ \theta^{(n)}],\ [(b'):\ \varphi'];\cdots;\ [(b)^{(n)}:\ \varphi^{(n)}]; \\ \\ [(c):\ \psi',\cdots,\ \psi^{(n)}],\ [(d'):\ \delta'];\cdots;\ [(d)^{(n)}:\ \delta^{(n)}];\end{matrix}\ x_1,\cdots,x_n\right]$$

$$= \sum_{m_1,\cdots,m_n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\cdots+m_n\theta_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots+m_n\psi_j^{(n)}}} \frac{\prod_{j=1}^{B'}(b'_j)_{m_1\varphi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\varphi_j^{(n)}}}{\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

$$\times \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}. \tag{16}$$

In (16) we use the following notation, generalizing the Pochhammer symbol (2):

$$(\gamma)_z = \frac{\Gamma(\gamma+z)}{\Gamma(\gamma)} \quad (\gamma,\ z\in\mathbb{C};\quad \gamma+z\neq 0,-1,-2,\cdots), \tag{17}$$

and the multiple series in (16) converges absolutely for all $x_1,\cdots,x_n$ for $\Delta_i > 0$ or for $\Delta_i = 0$ and $|z_i| < \varrho_i,\ i=1,\cdots,n$, where

$$\Delta_i = 1 + \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{D^{(i)}}\delta_j^{(i)} - \sum_{j=1}^{A}\theta_j^{(i)} - \sum_{j=1}^{B^{(i)}}\varphi_j^{(i)} \quad (i=1,\cdots,n),$$

$$E_i = (\mu_i)^{1+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\varphi_j^{(i)}} \frac{\prod_{j=1}^{C}\left(\sum_{i=1}^{n}\mu_i\psi_j^{(i)}\right)^{\psi_j^{(i)}}\prod_{j=1}^{D^{(i)}}(\delta_j^{(i)})^{\delta_j^{(i)}}}{\prod_{j=1}^{A}\left(\sum_{i=1}^{n}\mu_i\theta_j^{(i)}\right)^{\theta_j^{(i)}}\prod_{j=1}^{B^{(i)}}(\varphi_j^{(i)})^{\varphi_j^{(i)}}},$$

$$(i=1,\cdots,n), \quad \varrho_i = \min_{\mu_1,\cdots,\mu_n}\{E_i\} \quad (i=1,\cdots,n). \tag{18}$$

The idea for the above representations follows from the representation for (14) in terms of the fractional integral (15) and of the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$.

The paper is organized as follows. In Section 2 we give representation of (14) and (5) in terms of the Riemann-Liouville fractional integral (15). Section 3 is devoted to special cases involving the Mittag-Lefler function $E_{\alpha,\delta}$, the Kummer and the Gauss hypergeometric functions ${}_1F_1$ and ${}_2F_1$, the generalized hypergeometric functions ${}_2F_2$, the generalized Wright functions ${}_2\Psi_2$ and ${}_1\Psi_1$. In Section 4 we obtain the representation for the modification of (5) in terms of special function (16) and express the special cases in terms of the Mittag-Leffler function (3). Section 5 deals with the representation of (6) and its modification in terms of the Srivastava-Daoust function (16).

2. Representation of R_k and $M^{\gamma}_{\alpha,\beta}$ in terms of fractional integrals

In this section we deduce representations for the Dirichlet averages $R_k(\beta,\beta';x,y)$ and $M^{\gamma}_{\alpha,\delta}(\beta,\beta';x,y)$ via the fractional integral (15). The first assertion yields such a representation for the former.

Proposition 1. *Let $x,\ y$ be such that $y>x>0,\ \beta,\ \beta'>0$ and $k\in\mathbb{R}$. Let R_k and $I^{\beta'}_{0+}$ be given by (14) and (15), respectively. Then there hold the following formula,*

$$R_k(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}\ (y-x)^{1-\beta-\beta'}\ [I^{\beta'}_{0+}(t^{\beta-1}(y-t)^k)](y-x). \quad (19)$$

P r o o f. For $y>x>0$ rewrite $R_k(\beta,\beta';x,y)$ given by (14) in the form

$$R_k(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\int_0^1[y-u(y-x)]^k\ u^{\beta-1}(1-u)^{\beta'-1}du. \quad (20)$$

Making the change of variables $u(y-x)=t$, we have

$$R_k(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}(y-x)^{1-\beta-\beta'}\int_0^{y-x}(y-t)^k\ t^{\beta-1}(y-x-t)^{\beta'-1}dt.$$

According to (15), this proves (19). ∎

Let ${}_2F_1(a,b;c;z)$ be the Gauss hypergeometric function defined for complex parameters $a,b,c\in\mathbb{C}$ $(c\neq 0,-1,\cdots)$ by the hypergeometric series

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} \quad (21)$$

which converges absolutely for $|z|<1$ and $|z|=1,\ \Re(c-a-b)>0$; for example, see Erdélyi et al [4].

The next assertion yields the representation of $R_k(\beta,\beta';x,y)$ in terms of (21).

Proposition 2. *Let the condition of Proposition 1 be hold and let $|1-\frac{x}{y}|<1$. Then there holds the following representation*

$$R_k(\beta,\beta';x,y)=y^k\ {}_2F_1\left(-k,\beta;\beta+\beta';1-\frac{x}{y}\right). \quad (22)$$

P r o o f. Taking out y^k from the integrand in (20), we have

$$R_k(\beta,\beta';x,y) = y^k \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}$$

$$\times \int_0^1 \left[\left(1 - u\left(1 - \frac{x}{y}\right)\right)^{-(-k)} u^{\beta-1}(1-u)^{\beta'-1} du \right]. \tag{23}$$

According to the known integral representation for the Gauss hypergeometric function (21) (see [4]),

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} dx \tag{24}$$

$$(0 < \Re(b) < \Re(c); \quad |arg(1-z)| < \pi),$$

(23) yields (22). ■

A statement similar to Proposition 1 for the Mittag-Leffler Dirichlet average (5) is given by the following result.

THEOREM 1. *Let* $\alpha,\ \delta,\ \gamma,\ \beta,\ \beta' \in \mathbb{C}$ *be complex numbers,* $\Re(\alpha) > 0,\ \Re(\beta) > 0,\ \Re(\beta') > 0$*, and let* $x, y \in \mathbb{R}$ *be real numbers such that* $x > y$*, and let* $M^{\gamma}_{\alpha,\delta}$ *and* $I^{\beta'}_{0+}$ *be given by (5) and (14), respectively. Then the Dirichlet average of Mittag-Leffler function is given by the formula*

$$M^{\gamma}_{\alpha,\delta}(\beta,\beta';x,y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(x-y)^{1-\beta-\beta'}\ [I^{\beta'}_{0+}(t^{\beta-1}\ E^{\gamma}_{\alpha,\delta}(y+t))](x-y). \tag{25}$$

P r o o f. According to (5) and (1) we have

$$M^{\gamma}_{\alpha,\delta}(\beta,\beta';x,y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} \int_0^1 \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\delta) n!}$$

$$\times [ux + (1-u)y]^n\ u^{\beta-1}(1-u)^{\beta'-1} du \tag{26}$$

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} \int_0^1 \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\delta) n!} [y + u(x-y)]^n\ u^{\beta-1}(1-u)^{\beta'-1} du. \tag{27}$$

Changing the orders of integration and summation (which is possible by uniform convergence of the series in (27)), we find

$$M^{\gamma}_{\alpha,\delta}(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\delta)n!}$$

$$\times\int_0^1[y+u(x-y)]^n\ u^{\beta-1}(1-u)^{\beta'-1}du.$$

Changing the variable $u(x-y)=t$ and taking (15) into account, we have

$$M^{\gamma}_{\alpha,\delta}(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\delta)n!}(x-y)^{1-\beta-\beta'}$$

$$\times\int_0^{x-y}(y+t)^n t^{\beta-1}(x-y-t)^{\beta'-1}dt$$

$$=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}(x-y)^{1-\beta-\beta'}\int_0^{x-y}E^{\gamma}_{\alpha,\delta}(y+t)\ t^{\beta-1}(x-y-t)^{\beta'-1}dt$$

$$=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(x-y)^{1-\beta-\beta'}\ [I^{\beta'}_{0+}(t^{\beta-1}E^{\gamma}_{\alpha,\delta}(y+t))](x-y).$$

This yields (25), and thus the theorem is proved. ■

3. Special cases

In this section we consider some particular cases of Theorem 1. Setting $\alpha=1$ in this theorem, according to (4) we obtain the first result.

COROLLARY 1.1. *Let the conditions of Theorem 1 be valid and* $\alpha=1$. *Then*

$$M^{\gamma}_{1,\delta}(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\delta)}(x-y)^{1-\beta-\beta'}$$
$$\times[I^{\beta'}_{0+}(t^{\beta-1}\ {}_1F_1(\gamma;\delta;(y+t)))](x-y). \tag{28}$$

Taking $\gamma=1$ in Theorem 1 and using (3), we come to the second result.

COROLLARY 1.2. *Let the conditions of Theorem 1 be satisfied and* $\gamma=1$. *Then*

$$M^{1}_{\alpha,\delta}(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(x-y)^{1-\beta-\beta'}\ [I^{\beta'}_{0+}(t^{\beta-1}E_{\alpha,\delta}(y+t))](x-y). \tag{29}$$

For the next results, being special cases for $y = 0$ and $x = 0$, we need the so-called generalized Wright hypergeometric function ${}_p\Psi_q(z)$ defined for complex $a_i,\ b_j \in \mathbb{C}$ and real $A_i,\ B_j \in \mathbb{R}$ $(i = 1, \cdots, p;\ j = 1, \cdots, q)$ by the series

$$ {}_p\Psi_q\left[\begin{array}{c} (b_1, B_1), ..., (b_q, B_q) \\ (a_1, A_1), ..., (a_p, A_p) \end{array}\Big| z\right] = \sum_{n=0}^{\infty} \frac{\{\prod_{i=1}^{p} \Gamma(a_i + A_i n)\} z^n}{\{\prod_{j=1}^{q} \Gamma(b_j + B_j n)\} n!}. \tag{30} $$

This function was introduced by Wright [20] who investigated in [20] – [22] its asymptotic behavior for large values of argument z under the condition

$$ \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > -1. \tag{31} $$

Properties of ${}_p\Psi_q[z]$ were studied in [10]. In particular, it was proved that ${}_p\Psi_q[z]$ is an entire function of $z \in \mathbb{C}$ under condition (31).

Using (25) and (29) we obtain the following result.

Corollary 1.3. *Let the conditions of Theorem 1 be valid with $\alpha > 0$ and $y = 0$. Then*

$$ M^{\gamma}_{\alpha,\delta}(\beta, \beta'; x, 0) = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)\Gamma(\gamma)}\ {}_2\Psi_2\left[\begin{smallmatrix} (\gamma,\ 1),\ (\beta,1) \\ (\beta+\beta',\ 1),\ (\delta,\ \alpha) \end{smallmatrix}\Big| x\right] \quad (x > 0). \tag{32} $$

In particular, when $\beta + \beta' = \gamma$,

$$ M^{\gamma}_{\alpha,\delta}(\beta, \gamma - \beta; x, 0) = \frac{1}{\Gamma(\beta)}\ {}_1\Psi_1\left[\begin{smallmatrix} (\beta,1) \\ (\delta,\ \alpha) \end{smallmatrix}\Big| x\right] \quad (x > 0). \tag{33} $$

P r o o f. Using (26), changing the orders of integration and summation and applying formulas of connections between the Pochhammer symbol and the beta function with the gamma function [4]

$$ (z)_k = \frac{\Gamma(z + k)}{\Gamma(z)} \quad (z \in \mathbb{C},\ k \in \mathbb{N}_0), \quad B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \ (\alpha,\ \beta \in \mathbb{C}), \tag{34} $$

we have

$$ M^{\gamma}_{\alpha,\delta}(\beta, \beta'; x, 0) = \frac{\Gamma(\beta + \beta')}{\Gamma(\gamma)\Gamma(\beta')} \sum_{n=0}^{\infty} \frac{(\gamma)_n\ x^n}{\Gamma(\alpha n + \delta) n!} \int_0^1 u^{n+\beta-1}(1 - u)^{\beta'-1} du $$

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\gamma)\Gamma(\beta')} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\delta)} \frac{x^n}{n!} \frac{\Gamma(n+\beta)\Gamma(\beta')}{\Gamma(n+\beta+\beta')}$$

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)}{\Gamma(\alpha n+\delta)} \frac{x^n}{n!} \frac{\Gamma(n+\beta)}{\Gamma(n+\beta+\beta')}.$$

In accordance with (30) this gives (32), and (33) follows from (32). ∎

The next result is proved similarly to Corollary 1.3.

Corollary 1.4. *Let the conditions of Theorem 1 be hold with $\alpha > 0$ and $x = 0$. Then*

$$M^{\gamma}_{\alpha,\delta}(\beta,\beta';0,y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta')\Gamma(\gamma)}\ {}_2\Psi_2\left[\begin{smallmatrix}(\gamma,\ 1),\ (\beta',1)\\ (\beta+\beta',\ 1),\ (\delta,\ \alpha)\end{smallmatrix}\Big| y\right] \quad (y<0). \tag{35}$$

In particular, when $\beta+\beta' = \gamma$,

$$M^{\gamma}_{\alpha,\delta}(\beta,\gamma-\beta;0,y) = \frac{1}{\Gamma(\beta')}\ {}_1\Psi_1\left[\begin{smallmatrix}(\beta',1)\\ (\delta,\ \alpha)\end{smallmatrix}\Big| y\right] \quad (y<0). \tag{36}$$

Using (32) and (35) with $\alpha = 1$, we obtain the representations of $M^{\gamma}_{1,\delta}$ in terms of the generalized hypergeometric function ${}_pF_q(z)$ defined for complex $a_1,\cdots,a_p,\ b_1,\cdots,b_q \in \mathbb{C}$ ($b_j \neq 0,-1,-2,\cdots;\ j=1,\cdots,q$) by the generalized hypergeometric series [4]

$${}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}. \tag{37}$$

This series is absolutely convergent for any $z \in C$ when $p \leq q$.

Corollary 1.5. *Let the conditions of Corollaries 1.3 and 1.4 be valid and let $\alpha = 1$. Then*

$$M^{\gamma}_{1,\delta}(\beta,\beta';x,0) = \frac{1}{\Gamma(\delta)}\ {}_2F_2(\gamma,\beta;\beta+\beta',\delta;x) \quad (x>0), \tag{38}$$

$$M^{\gamma}_{1,\delta}(\beta,\beta';0,y) = \frac{1}{\Gamma(\delta)}\ {}_2F_2(\gamma,\beta';\beta+\beta',\delta;y) \quad (y<0). \tag{39}$$

Setting $\alpha = \gamma = 1$, $\beta' = 1-\beta$, from (38) and (39) we obtain the following assertion.

COROLLARY 1.6. *Let the conditions of Corollary 1.5 be satisfied and let $\alpha = 1$ and $\beta' = 1 - \beta$. Then*

$$M^{1}_{1,\delta}(\beta, 1-\beta; x, 0) = \frac{1}{\Gamma(\delta)}\ {}_1F_1(\beta; \delta; x) \quad (x > 0), \tag{40}$$

$$M^{1}_{1,\delta}(\beta, 1-\beta; 0, y) = \frac{1}{\Gamma(\delta)}\ {}_1F_1(\beta'; \delta; y) \quad (y < 0). \tag{41}$$

4. Representation for modification of $M^{\gamma;\alpha}_{\alpha,\delta}(\beta, \beta'; x, y)$ in terms of Srivastava-Daoust function

In this section we consider the Dirichlet average, being a modification of the one in (5), in the form

$${}_\rho M^{\gamma,\alpha}_{\alpha,\delta}(\beta, \beta'; x, y) = \int_{E_2} (u \circ z)^{\rho-1} E^{\gamma}_{\alpha,\beta}[(u \circ z)^{\alpha}] d\mu_{\beta,\beta'}(u). \tag{42}$$

There holds the following analogue of Theorem 1 giving representation for (42).

THEOREM 2. *Let $\alpha,\ \delta,\ \gamma,\ \rho,\ \beta,\ \beta' \in \mathbb{C}$ be complex numbers, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\beta') > 0$, and let $x, y \in \mathbb{R}$ be real numbers such that $x > y$. Then there holds the following relation:*

$${}_\rho M^{\gamma,\alpha}_{\alpha,\delta}(\beta, \beta'; x, y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(x-y)^{1-\beta-\beta'}$$
$$\times [I^{\beta'}_{y+}\ (t^{\rho-1}(t-y)^{\beta-1} E^{\gamma}_{\alpha,\delta}(t^{\alpha}))](x). \tag{43}$$

P r o o f. Using (42) and (11) and changing the orders of integration and summation, we have

$${}_\rho M^{\gamma,\alpha}_{\alpha,\delta}(\beta, \beta'; x, y) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} \int_0^1\ [ux + (1-u)y]^{\rho-1} E^{\gamma}_{\alpha,\delta}([ux + (1-u)y]^{\alpha})$$
$$\times u^{\beta-1}(1-u)^{\beta'-1} du$$
$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\delta) n!}\ \int_0^1 [y + u(x-y)]^{\alpha n+\rho-1} u^{\beta-1}(1-u)^{\beta'-1} du.$$

Making the change $y+u(x-y)=t$ and applying again (1), we obtain

$$
{}_\rho M_{\alpha,\delta}^{\gamma,\alpha}(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}(x-y)^{1-\beta-\beta'}
\times\int_y^x t^{\rho-1}(t-y)^{\beta-1}E_{\alpha,\delta}^{\gamma}(t^\alpha)(x-t)^{\beta'-1}dt.
$$

In accordance with (15), this yields the result in (43). ■

Corollary 2.1. *Let the conditions of Theorem 2 be valid with $\gamma=1$. Then*

$$
{}_\rho M_{\alpha,\delta}^{1,\alpha}(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(x-y)^{1-\beta-\beta'}
\times[I_{y+}^{\beta'}\ (t^{\rho-1}(t-y)^{\beta-1}E_{\alpha,\delta}^{1}(t^\alpha))](x)\quad(x>y). \tag{44}
$$

Setting $\gamma=1,\ \delta=\beta+\rho-1$ and $y=0$ in Corollary 2.1, we have

Corollary 2.2. *Let the conditions of Corollary 2.1 be satisfied, $\delta=\beta+\rho-1$, and let $x>0$ and $y=0$. Then*

$$
{}_\rho M_{\alpha,\beta+\rho-1}^{1,\alpha}(\beta,\beta';x,0)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}x^{\rho-1}E_{\alpha,\beta+\rho+\beta'-1}(x^\alpha)\quad(x>0). \tag{45}
$$

In particular if $\beta'=1-\beta$, then

$$
{}_\rho M_{\alpha,\beta+\rho-1}^{1,\alpha}(\beta,1-\beta;x,0)=\frac{1}{\Gamma(\beta)}x^{\rho-1}E_{\alpha,\rho}(x^\alpha)\quad(x>0). \tag{46}
$$

P r o o f. Setting $\delta=\beta+\rho-1$ and $y=0$ in (44), using (15) and changing the orders of integration and summation, we have

$$
\begin{aligned}
{}_\rho M_{\alpha,\beta+\rho-1}^{1,\alpha}(\beta,\beta';x,0)&=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}x^{1-\beta-\beta'}\ [I_{y+}^{\beta'}\ (t^{\beta+\rho-1}E_{\alpha,\beta+\rho-1}(t^\alpha))](x)\\
&=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}x^{1-\beta-\beta'}\int_0^x(x-t)^{\beta'-1}\ t^{\beta+\rho-2}\sum_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+\beta+\rho-1)}dt\\
&=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}x^{1-\beta-\beta'}x^{\beta'-1}\ \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta+\rho-1)}\\
&\qquad\times\int_0^x(1-\frac{t}{x})^{\beta'-1}\ t^{\alpha k+\beta+\rho-2}dt.
\end{aligned}
$$

Making substitution $t = ux$ and using the second relation in (34), we find

$$ {}_{\rho}M^{1,\alpha}_{\alpha,\beta+\rho-1}(\beta,\beta';x,0) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)} x^{-\beta} \sum_{k=0}^{\infty} \frac{x^{\alpha k+\beta+\rho-1}}{\Gamma(\alpha k+\beta+\rho+\beta'-1)}. $$

By (3), from here we deduce the result in (45). (46) follows from (45). Thus the theorem is proved. ■

COROLLARY 2.3. *Let the conditions of Corollary 2.1 be satisfied,* $\rho = 1$, $\delta = \beta$, *and let* $x > 0$ *and* $y = 0$. *Then*

$$ {}_{1}M^{1,\alpha}_{\alpha,\beta}(\beta,\beta';x,0) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)} E_{\alpha,\beta+\beta'}(x^{\alpha}) \quad (x>0). \tag{47} $$

In particular, if $\beta' = 1-\beta$, *then*

$$ {}_{1}M^{1,\alpha}_{\alpha,\beta}(\beta,1-\beta;x,0) = \frac{1}{\Gamma(\beta)} E_{\alpha}(x^{\alpha}) \quad (x>0), \tag{48} $$

where

$$ E_{\alpha}(z) = E_{\alpha,1}(z). \tag{49} $$

The proof is similar to the proof of Corollary 2.2.

In conclusion of this section, we consider a modification of (5) in the form

$$ {}_{\rho}M^{\gamma,\alpha}_{\alpha,\delta}(\beta,\beta';x,y) = \int_{E} (u \circ z)^{\rho-1} E^{\gamma}_{\alpha,\beta}[(u \circ z)^{\alpha}] d\mu_b(u). \tag{50} $$

The next statement yields the representation for (50) with $\rho = \delta$.

THEOREM 3. *Let* $\alpha, \delta, \gamma, \beta, \beta' \in \mathbb{C}$ *be complex numbers,* $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\beta') > 0$, *and let* $x, y \in \mathbb{R}$ *be real numbers. Then there holds the following relation:*

$$ {}_{\delta}M^{\gamma,\alpha}_{\alpha,\delta}(\beta,\beta';x,y) = \frac{y^{\delta-1}}{\Gamma(\delta)} F^{1:\ 1;1}_{0:\ 2;1} \left[\begin{matrix} [1-\delta:-\alpha;1]:\ [\gamma:1];\ [\beta:1]; \\ -:\ [1-\delta:-\alpha],\ [\delta,\alpha];\ [\beta+\beta':1]; \end{matrix} y^{\alpha}, 1-\frac{x}{y} \right], \tag{51} $$

where $F(\cdot)$ *is the Srivastava-Daoust function of the form (16).*

P r o o f. Using (50), changing the orders of integration and summation and applying the integral representation (23), we have

$$ {}_{\delta}M^{\gamma,\alpha}_{\alpha,\delta}(\beta,\beta';x,y) $$

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} \int_0^1 \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\delta)k!} [ux+(1-u)y]^{\alpha k+\delta-1} u^{\beta-1}(1-u)^{\beta'-1} du$$

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} \int_0^1 \sum_{k=0}^{\infty} \frac{(\gamma)_k \; y^{\alpha k+\delta-1}}{\Gamma(\alpha k+\delta)k!} [1-u(1-\frac{x}{y})]^{\alpha k+\delta-1} u^{\beta-1}(1-u)^{\beta'-1} du$$

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} \sum_{k=0}^{\infty} \frac{(\gamma)_k \; y^{\alpha k+\delta-1}}{\Gamma(\alpha k+\delta)k!} \frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(\beta+\beta')} \; {}_2F_1(\beta, -\alpha k-\delta+1; \beta+\beta'; 1-\frac{x}{y}).$$

Applying (21), we obtain

$$ {}_\delta M^{\gamma,\alpha}_{\alpha,\delta}(\beta,\beta'; x,y) = y^{\delta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k y^{\alpha k}}{\Gamma(\alpha k+\delta)k!} \sum_{n=0}^{\infty} \frac{(\beta)_n(-(\alpha k+\delta-1))_n}{(\beta+\beta')_n} \times \frac{(1-\frac{x}{y})^n}{n!}.$$

By (17),

$$(-(\alpha k+\delta-1))_n = \frac{\Gamma(1-\delta-\alpha k+n)}{\Gamma(1-\delta-\alpha k)} = \frac{(1-\delta)_{-\alpha k+n}}{(1-\delta)_{-\alpha k}}, \Gamma(\alpha k+\delta) = \Gamma(\delta)(\delta)_{\alpha k},$$

and thus

$$ {}_\delta M^{\gamma,\alpha}_{\alpha,\delta}(\beta,\beta'; x,y) = \frac{y^{\delta-1}}{\Gamma(\delta)} \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\gamma)_k(\beta)_n(1-\delta)_{n-\alpha k}}{(\beta+\beta')_n(1-\delta)_{-\alpha k}} \frac{y^{\alpha k}}{k!} \frac{(1-\frac{x}{y})^n}{n!}.$$

From here, in accordance with (16), we obtain (51). This completes the proof of the theorem. ∎

5. Dirichlet average of multivariate function

In this section we consider the Dirichlet average (6) and its modification, where $z_1, \cdots, z_n$ are variables and $b_1, \cdots, b_n$ are parameters. Our results are based on the following preliminary assertion.

LEMMA 1. *Let $n \in \mathbb{C}$, let $b_i, r_i \in \mathbb{C}$ be complex numbers such that $\Re(b_i) > 0$, $\Re(r_i) > -1$ $(i = 1, \cdots, n)$, let E_{n-1} be the simplex (7) and let $d\mu_b(u)$ be given by (9). Then there holds the formula*

$$\int_{E_{n-1}} u_1^{r_1} \cdots u_{n-1}^{r_{n-1}} (1-u_1-\cdots-u_{n-1})^{r_n} d\mu_b(u) = \frac{(b_1)_{r_1}\cdots(b_n)_{r_n}}{(b_1+\cdots+b_n)_{r_1+\cdots+r_n}}. \tag{52}$$

In particular, if $r_1 = \cdots = r_n = 0$, then

$$\int_{E_{n-1}} d\mu_b(u) = 1. \tag{53}$$

P r o o f. Using (7) and (9)-(10), we have

$$\int_{E_{n-1}} u_1^{r_1} \cdots u_{n-1}^{r_{n-1}} (1 - u_1 - \cdots - u_{n-1})^{r_n} d\mu_b(u) = \frac{1}{B(b)} \int_0^1 u_1^{b_1+r_1-1} du_1$$

$$\int_0^{1-u_1} u_2^{b_2+r_2-1} du_2 \cdots \int_0^{1-u_1-\cdots-u_{n-3}} u_{n-2}^{b_{n-2}+r_{n-2}-1} du_{n-2}$$

$$\times \int_0^{1-u_1-\cdots-u_{n-2}} u_{n-1}^{b_{n-1}+r_{n-1}-1} (1 - u_1 - \cdots - u_{n-1})^{b_n+r_n-1} du_{n-1}.$$

Making the change of variable $u_{n-1} = s_{n-1}(1 - u_1 - \cdots - u_{n-2})$ in the last inner integral and using the second relation in (34), we have

$$\int_{E_{n-1}} u_1^{r_1} \cdots u_{n-1}^{r_{n-1}} (1 - u_1 - \cdots - u_{n-1})^{r_n} d\mu_b(u) = \frac{1}{B(b)} \int_0^1 u_1^{b_1+r_1-1} du_1$$

$$\times \int_0^{1-u_1} u_2^{b_2+r_2-1} \cdots \int_0^{1-u_1-\cdots-u_{n-3}} u_{n-2}^{b_{n-2}+r_{n-2}-1}$$

$$\times (1 - u_1 - \cdots - u_{n-2})^{b_{n-1}+b_n+r_{n-1}+r_n-1} du_{n-2}$$

$$\times \int_0^1 s_{n-1}^{b_{n-1}+r_{n-1}-1} (1 - s_{n-1})^{b_n+r_n-1} ds_{n-1}$$

$$= \frac{1}{B(b)} \frac{\Gamma(b_{n-1} + r_{n-1})\Gamma(b_n + r_n)}{\Gamma(b_{n-1} + b_n + r_{n-1} + r_n)} \int_0^1 u_1^{b_1+r_1-1} du_1 \int_0^{1-u_1} u_2^{b_2+r_2-1}$$

$$\cdots \int_0^{1-u_1-\cdots-u_{n-3}} u_{n-2}^{b_{n-2}+r_{n-2}-1} (1 - u_1 - \cdots - u_{n-2})^{b_{n-1}+b_n+r_{n-1}+r_n-1} du_{n-2}.$$

Continuing this process, we deduce the relation

$$\int_{E_{n-1}} u_1^{r_1} \cdots u_{n-1}^{r_{n-1}} (1 - u_1 - \cdots - u_{n-1})^{r_n} d\mu_b(u)$$

$$= \frac{\Gamma(b_1 + r_1) \cdots \Gamma(b_n + r_n)}{\Gamma(b_1) \cdots \Gamma(b_n)} \frac{\Gamma(b_1 + \cdots + b_n)}{\Gamma(b_1 + r_1 \cdots + b_n + r_n)}.$$

From here, in accordance with (17), we have the result in (52). By (2) with $k = 0$, (53) clearly follows from (52). ■

Using Lemma 1, we obtain the representation for $M_{\delta,\beta}^{\gamma,\alpha}(b_1, \cdots, b_n; 1 - z_1, \cdots, 1 - z_n)$.

THEOREM 4. *Let $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{C}$, $\Re(\beta) > 0$, $\Re(\delta) > 0$, and let $b_i,\ z_i \in \mathbb{C}$ $\Re(b_i) > 0$ $(i = 1, \cdots, n)$ be complex numbers. Then there holds the following formula*

$$M_{\delta,\beta}^{\gamma,\alpha}(b_1, \cdots, b_n; 1 - z_1, \cdots, 1 - z_n) = \frac{1}{\Gamma(\beta)} + \frac{\gamma}{\Gamma(\delta + \beta)}$$

$$\times F_{2:\ 2;0;\cdots;0}^{1:\ 2;1;\cdots;1}\left(\begin{matrix}[-\alpha{:}-\alpha,1,\cdots,1]{:}\ [\gamma+1{:}1],\ [1{:}1];\ [b_1{:}1];\ \cdots;\ [b_n{:}1];\\ [-\alpha{:}\alpha,0,\cdots,0];[b_1+\cdots+b_n{:}0,1,\cdots,1]{:}\ [\delta+\beta{:}\delta],\ [2{:}1];-;\end{matrix}1, z_1, \cdots, z_n\right), \quad (54)$$

where $F(\cdot)$ is the Srivastava-Daoust function of the form (16).

P r o o f. By (6) for $\Re(\beta) > 0$, $\Re(\delta) > 0$, in accordance with (6) and (1) we have

$$M_{\delta,\beta}^{\gamma,\alpha}(b_1, \cdots, b_n; 1 - z_1, \cdots, 1 - z_n)$$

$$= \int_{E_{n-1}} \left[\sum_{k=0}^{\infty} \frac{(\gamma)_k (1 - u \circ z)^{\alpha k}}{\Gamma(\delta k + \beta)\ k!}\right] d\mu_b(u)$$

$$= \int_{E_{n-1}} \left[\sum_{k=0}^{\infty} \frac{(\gamma)_k (1 - [\sum_{i=1}^{n-1} u_i z_i + (1 - u_1 - \cdots - u_{n-1}) z_n])^{\alpha k}}{\Gamma(\delta k + \beta)\ k!}\right] d\mu_b(u).$$

Applying the directly verified formula

$$(1 - x_1 - \cdots - x_n)^{\alpha} = \sum_{r_1, \cdots, r_n = 0}^{\infty} (-\alpha)_{r_1 + \cdots + r_n} \frac{x_1^{r_1} \cdots x_n^{r_n}}{r_1! \cdots r_n!} \quad (|x_1 + \cdots + x_n| < 1), \quad (55)$$

changing the orders of integration and summation, we find

$$M_{\delta,\beta}^{\gamma,\alpha}(b_1, \cdots, b_n; 1 - z_1, \cdots, 1 - z_n)$$

$$= \int_{E_{n-1}} \left[\sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\delta k + \beta)\ k!} \sum_{r_1, \cdots, r_n = 0}^{\infty} (-\alpha k)_{r_1 + \cdots + r_n}\right.$$

$$\left. \times \frac{(u_1 z_1)^{r_1} \cdots (u_{n-1} z_{n-1})^{r_{n-1}} [(1 - u_1 - \cdots - u_{n-1}) z_n]^{r_n}}{r_1! \cdots r_n!}\right] d\mu_b(u)$$

$$= \frac{1}{\Gamma(\beta)} \int_{E_{n-1}} d\mu_b(u) + \sum_{k=1}^{\infty} \frac{(\gamma)_k}{\Gamma(\delta k+\beta)\ k!} \sum_{r_1,\cdots,r_n=0}^{\infty} (-\alpha k)_{r_1+\cdots+r_n}$$

$$\times \frac{(z_1)^{r_1}\cdots(z_n)^{r_n}}{r_1!\cdots r_n!} \int_{E_{n-1}} (u_1)^{r_1}\cdots(u_{n-1})^{r_{n-1}}[(1-u_1-\cdots-u_{n-1})]^{r_n} d\mu_b(u).$$

Using (52) and (53) and taking (17) into account, we obtain

$$M_{\delta,\beta}^{\gamma,\alpha}(b_1,\cdots,b_n;1-z_1,\cdots,1-z_n) = \frac{1}{\Gamma(\beta)} + \sum_{k=1}^{\infty} \frac{(\gamma)_k}{\Gamma(\delta k+\beta)\ k!}$$

$$\times \sum_{r_1,\cdots,r_n=0}^{\infty} \frac{(-\alpha k)_{r_1+\cdots+r_n}(b_1)_{r_1}\cdots(b_n)_{r_n}}{(b_1+\cdots+b_n)_{r_1+\cdots+r_n}} \frac{(z_1)^{r_1}\cdots(z_n)^{r_n}}{r_1!\cdots r_n!}$$

$$= \frac{1}{\Gamma(\beta)} + \frac{\gamma}{\Gamma(\beta+\delta)} \sum_{k=0}^{\infty} \frac{(\gamma+1)_k(1)_k}{(\delta+\beta)_{\delta k}(2)_k k!}$$

$$\times \sum_{r_1,\cdots,r_n=0}^{\infty} \frac{(-\alpha k-\alpha)_{r_1+\cdots+r_n}(b_1)_{r_1}\cdots(b_n)_{r_n}}{(b_1+\cdots+b_n)_{r_1+\cdots+r_n}} \frac{(z_1)^{r_1}\cdots(z_n)^{r_n}}{r_1!\cdots r_n!}$$

$$= \frac{1}{\Gamma(\beta)} + \frac{\gamma}{\Gamma(\delta+\beta)} \sum_{k=0}^{\infty} \frac{(\gamma+1)_k\ (1)_k}{(\delta+\beta)_{\delta k}\ (2)_k\ k!}$$

$$\times \sum_{r_1,\cdots,r_n=0}^{\infty} \frac{(-\alpha)_{-\alpha k+r_1+\cdots+r_n}(b_1)_{r_1}\cdots(b_n)_{r_n}}{(-\alpha)_{-\alpha k}(b_1+\cdots+b_n)_{r_1+\cdots+r_n}} \frac{(z_1)^{r_1}\cdots(z_n)^{r_n}}{r_1!\cdots r_n!}.$$

In accordance with (16), this yields (54). Note that the Srivastava-Daoust function in (54) exists because the last series is absolutely convergent, see (16) and (18). This completes the proof of Theorem 4. ■

For the next result we need the Lauricella function defined for complex $a, b_1, \cdots, b_n, c \in \mathbb{C}$ and for complex $z_1, \cdots, z_n \in \mathbb{C}$, with $|z_1| < 1, \cdots, |z_n| < 1$, by the multidimensional series [1], [4]

$$F_D(a, b_1, \cdots, b_n; c; z_1, \cdots, z_n)$$

$$= \sum_{m_1,\cdots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{{z_1}^{m_1}\cdots{z_n}^{m_n}}{m_1!\cdots m_n!}. \tag{56}$$

Consider the following Dirichlet average

$$ {}_{\rho}M^{\gamma}_{\delta,\beta}(b_1,\cdots,b_n;1-z_1,\cdots,1-z_n)=\int_{E_{n-1}}(1-u\circ z)^{\rho-1}E^{\gamma}_{\delta,\beta}[(1-u\circ z)^{\delta}]d\mu_b(u). \tag{57} $$

The following statement gives a representation of (57) for $\rho=\beta$ in terms of the Srivastava-Daoust function (16).

THEOREM 5. *Let $\beta,\ \gamma,\ \delta\in\mathbb{C}$, $\Re(\beta)>0,\ \Re(\delta)>0$, and let $b_1,\cdots,b_n\in\mathbb{C}$ be complex numbers, and let $z_1,\cdots,z_n\in\mathbb{C}$ be complex variables. Then*

$$ {}_{\beta}M^{\gamma}_{\delta,\beta}(b_1,\cdots,b_n;1-z_1,\cdots,1-z_n) $$

$$ =\frac{1}{\Gamma(\beta)}\ F^{1:\ 1;1;\cdots;1}_{2:\ 0;0;\cdots;0}\left(\begin{matrix}[\beta;\delta;0;\cdots;0]:\ [\gamma{:}1];[b_1{:}1];\cdots;[b_n{:}1];\\ [b_1+\cdots+b_n{:}0;1;\cdots;1],\ [\beta{:}\delta;-1;\cdots;-1]:\ -;-;\cdots;-;\end{matrix}1,-z_1,\cdots,-z_n\right). \tag{58} $$

P r o o f. Using (57), applying (55) and (52), changing the orders of integration and summation, evaluating the obtained integrals and taking (56) into account, we have for $|u_1z_1+\cdots+u_nz_n|<1$:

$$ {}_{\beta}M^{\gamma}_{\delta,\beta}(b_1,\cdots,b_n;1-z_1,\cdots,1-z_n) $$

$$ =\int_{E_{n-1}}\sum_{k=0}^{\infty}\frac{(\gamma)_k(1-u\circ z)^{\delta k+\beta-1}}{\Gamma(\delta k+\beta)\ k!}d\mu_b(u) $$

$$ =\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\delta k+\beta)}\sum_{r_1,\cdots,r_n=0}^{\infty}(-\beta-k\delta+1)_{r_1+\cdots+r_n}\frac{z_1^{r_1}\cdots z_n^{r_n}}{r_1!\cdots r_n!} $$

$$ \times\int_{E_{n-1}}u_1^{r_1}\cdots u_{n-1}^{r_{n-1}}(1-u_1-\cdots u_{n-1})^{r_n}d\mu_b(u) $$

$$ =\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\delta k+\beta)}\sum_{r_1,\cdots,r_n=0}^{\infty}\frac{(-\beta-k\delta+1)_{r_1+\cdots+r_n}}{(b_1+\cdots+b_n)_{r_1+\cdots+r_n}}(b_1)_{r_1}\cdots(b_n)_{r_n}\frac{z_1^{r_1}\cdots z_n^{r_n}}{r_1!\cdots r_n!} $$

$$ =\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\delta k+\beta)k!}\ F_D(-\delta k-\beta+1,b_1,\cdots,b_n;b_1+\cdots+b_n;z_1,\cdots,z_n), \tag{59} $$

where F_D is the Lauricella function (56).

Since

$$ (-\beta-k\delta+1)_{r_1+\cdots+r_n}=(-1)^{r_1+\cdots r_n}(\beta+\delta k-1)\cdots(\beta+\delta k-r_1-\cdots-r_n) $$

$$= (-1)^{r_1+\cdots+r_n} \frac{\Gamma(\beta+\delta k)}{\Gamma(\beta+\delta k - r_1 - \cdots - r_n)} = (-1)^{r_1+\cdots+r_n} \frac{(\beta)_{\delta k}}{(\beta)_{\delta k - r_1 - \cdots - r_n}},$$

then form (59) we have

$$ {}_\beta M^\gamma_{\delta,\beta}(b_1,\cdots,b_n;1-z_1,\cdots,1-z_n)$$

$$= \sum_{k=0}^\infty \sum_{r_1,\cdots,r_n=0}^\infty \frac{(\gamma)_k (b_1)_{r_1}\cdots(b_n)_{r_n}}{(b_1+\cdots+b_n)_{r_1+\cdots+r_n}\Gamma(\delta k+\beta)}$$

$$\times \frac{(-1)^{r_1+\cdots+r_n}\Gamma(\delta k+\beta)}{\Gamma(\delta k+\beta-(r_1+\cdots+r_n))} \frac{1^k z_1^{r_1}\cdots z_n^{r_n}}{k! r_1!\cdots r_n!}$$

$$= \sum_{k=0}^\infty \sum_{r_1,\cdots,r_n=0}^\infty \frac{(\gamma)_k (b_1)_{r_1}\cdots(b_n)_{r_n}(\beta)_{\delta k}}{(b_1+\cdots+b_n)_{r_1+\cdots+r_n}(\beta)_{\delta k-r_1-\cdots-r_n}} \frac{1^k\ (-z_1)^{r_1}\cdots(-z_n)^{r_n}}{k!\ r_1!\cdots r_n!}.$$

According to (16), from here we arrive at (58), and thus the theorem is proved. ∎

Corollary 5.1. *Let the conditions of Theorem 5 be satisfied and let* $z_1 = \cdots = z_n = z,\ |z| < 1$. *Then*

$$ {}_\beta M^\gamma_{\delta,\beta}(b_1,\cdots,b_n;1-z,\cdots,1-z) = (1-z)^{\beta-1}\ E^\gamma_{\delta,\beta}[(1-z)^\delta]. \tag{60}$$

P r o o f. Taking $z_1 = \cdots = z_n = z$, with $|z| < 1$ in (59), and using the formula [19]:

$$F_D(a,b_1,\cdots,b_n;c;z,\cdots,z) = {}_2F_1(a,b_1+\cdots+b_n;c;z) \tag{61}$$

and (1), we have

$$ {}_\beta M^\gamma_{\delta,\beta}(b_1,\cdots,b_n;1-z,\cdots,1-z)$$

$$= \sum_{k=0}^\infty \frac{(\gamma)_k}{\Gamma(\delta k+\beta)k!}\ {}_2F_1(-\delta k-\beta+1,b_1+\cdots+b_n;b_1+\cdots+b_n;z)$$

$$= \sum_{k=0}^\infty \frac{(\gamma)_k}{\Gamma(\delta k+\beta)k!}\ {}_1F_0(-\delta k-\beta+1;\ ;z)$$

$$= \sum_{k=0}^\infty \frac{(\gamma)_k}{\Gamma(\delta k+\beta)k!}\ (1-z)^{\delta k+\beta-1}$$

$$= (1-z)^{\beta-1} \sum_{k=0}^\infty \frac{(\gamma)_k}{\Gamma(\delta k+\beta)k!}\ (1-z)^{\delta k} = (1-z)^{\beta-1}\ E^\gamma_{\delta,\beta}[(1-z)^\delta].$$

∎

Finally we consider the Dirichlet average of the form

$$M_{\delta,\beta}^{\gamma}(b_1,\cdots,b_n;z_1,\cdots,z_n)=\int_{E_{n-1}}E_{\delta,\beta}^{\gamma}(u\circ z)d\mu_b(u). \tag{62}$$

There holds the following result, being an analogue of Theorems 4 and 5.

THEOREM 6. *Let the conditions of Theorem 5 be satisfied. Then*

$$M_{\delta,\beta}^{\gamma}(b_1,\cdots,b_n;z_1,\cdots,z_n)$$

$$=\frac{1}{\Gamma(\beta)}F_{2:\ 0;\cdots;0}^{1:\ 1;\cdots;1}\left({[\gamma:1,\cdots,1]:\ [b_1:1];\ \cdots;\ [b_n:1]; \atop [\beta:\delta,\cdots,\delta],\ [b_1+\cdots+b_n:1,\cdots,1]:\ -;\cdots;-;}z_1,\cdots,z_n\right). \tag{63}$$

P r o o f. By (62), (1) and (8), we have

$$M_{\delta,\beta}^{\gamma}(b_1,\cdots,b_n;z_1,\cdots,z_n)=\int_{E_{n-1}}E_{\delta,\beta}^{\gamma}(u\circ z)d\mu_b(u)$$

$$=\int_{E_{n-1}}\sum_{k=0}^{\infty}\frac{(\gamma)_k(u_1z_1+\cdots+u_{n-1}z_{n-1}+(1-u_1-\cdots-u_{n-1})z_n)^k}{\Gamma(\beta)(\beta)_{\delta k}\ k!}d\mu_b(u).$$

Applying the formula from [17]:

$$\sum_{k=0}^{\infty}\frac{f(k)(x_1+\cdots+x_n)^k}{k!}=\sum_{r_1,\cdots,r_n=0}^{\infty}\frac{f(r_1+\cdots+r_n)x_1^{r_1}\cdots x_n^{r_n}}{r_1!\cdots r_n!}, \tag{64}$$

we find

$$M_{\delta,\beta}^{\gamma}(b_1,\cdots,b_n;z_1,\cdots,z_n)$$

$$=\int_{E_{n-1}}\left[\frac{1}{\Gamma(\beta)}\sum_{r_1,\cdots,r_n=0}^{\infty}\frac{(\gamma)_{r_1+\cdots+r_n}}{(\beta)_{\delta(r_1+\cdots+r_n)}}\right.$$

$$\left.\times\frac{(z_1u_1)^{r_1}\cdots(z_{n-1}u_{n-1})^{r_{n-1}}[z_n(1-u_1-\cdots-u_{n-1})]^{r_n}}{r_1!\cdots r_n!}\right]d\mu_b(u)$$

$$=\frac{1}{\Gamma(\beta)}\frac{1}{B(b)}\int_{E_{n-1}}\sum_{r_1,\cdots,r_n=0}^{\infty}\frac{(\gamma)_{r_1+\cdots+r_n}}{(\beta)_{\delta(r_1+\cdots+r_n)}}\frac{(z_1)^{r_1}\cdots(z_n)^{r_n}}{r_1!\cdots r_n!}$$

$$\times u_1^{r_1+b_1-1}\cdots u_{n-1}^{r_{n-1}+b_{n-1}-1}(1-u_1-\cdots-u_{n-1})^{r_n+b_n-1}du_1\cdots du_{n-1}.$$

Using the same arguments as in the proof of Theorem 5 and taking (15) into account, we have

$$M^{\gamma}_{\delta,\beta}(b_1,\cdots,b_n;z_1,\cdots,z_n)$$

$$= \frac{1}{\Gamma(\beta)} \sum_{r_1,\cdots,r_n=0}^{\infty} \frac{(\gamma)_{r_1+\cdots+r_n}}{(\beta)_{\delta(r_1+\cdots+r_n)}} \frac{(z_1)^{r_1}\cdots(z_n)^{r_n}}{r_1!\cdots r_n!} \frac{(b_1)_{r_1}\cdots(b_n)_{r_n}}{(b_1+\cdots+b_n)_{r_1+\cdots+r_n}}$$

$$= \frac{1}{\Gamma(\beta)} F^{1:\ 1;\cdots;1}_{2:\ 0;\cdots;0}\left({}^{[\gamma:1,\cdots,1]:\ [b_1:1];\ \cdots;\ [b_n:1];}_{[\beta:\delta,\cdots,\delta],\ [b_1+\cdots+b_n:1,\cdots,1]:\ -;\cdots;-;} \ z_1,\cdots,z_n \right).$$

This yields (63), which completes the proof of the theorem. ■

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[1] *Faculty of Mathematics and Mechanics*
Belarusian State University
220050 Minsk, BELARUS

e-mail: anatolykilbas@gmail.com

[2] *Centre for Mathematical Sciences, Pala Campus*
Arunapuram P.O, Palai,
Kerala 686 574, INDIA

e-mail: anithawilson82@gmail.com; cmspala@gmail.com